Technical appendix to:

Wealth Accumulation over the Life-Cycle and Precautionary Savings

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1 The solution method

To solve the model, I apply the method proposed by Deaton [1991] and often used in models of life cycle behavior with uncertainty.

Defining X_t= cash in hand = $R\ W_t + Y_t + B_t$, with evolution

$$X_{t+1} = R(X_t - C_t) + Y_{t+1} + B_{t+1} \tag{1}$$

the first order conditions of the problem are

$$C_t^{-\gamma} \geq \beta e^{\theta \Delta Z_{t+1}}\ \pi_{t+1} E_t(R\ C_{t+1}^{-\gamma}) + (1 - \pi_{t+1})\beta\alpha(X_t - C_t)^{-\gamma}$$

$$C_t \leq X_t$$

where the Euler equation is satisfied with equality if the borrowing constraint is not binding ($C_t < X_t$). π_{t+1} is the probability of being alive at $t+1$ conditional on being alive at t, and the expected value is conditional on surviving up to $t+1$. One can divide the conditions above by current earnings Y_t (lowercase letters will represent ratios to current earnings), and solve the problem backwards, starting from the last period T (after which the household dies with probability one), when

$$c_T = \frac{(\alpha\beta)^{-\frac{1}{\gamma}}}{1 + (\alpha\beta)^{-\frac{1}{\gamma}}} x_T \tag{2}$$

and then, working backwards, for every period t:

$$c_t^{-\gamma} = \beta e^{\Delta Z_{t+1}\theta}\ \pi_{t+1} E_t(R c_{t+1}^{-\gamma} g_{Yt}^{-\gamma}) + (1-\pi_{t+1})\beta\alpha(x_t - c_t)^{-\gamma} \tag{3}$$

where g_{Yt} is the rate of growth of income between period t and $t+1$. Equation (3) shows that one can use as state variables x, the current cash in hand to income ratio, and η, the innovation to the rate of growth of income (the latter variable is needed to evaluate the expected value). Note that the level of earnings Y is not a state variable anymore. This simplifies the problem and speeds up the computations dramatically.

I create a grid of points for the state variables x and η, and solve numerically the implicit equation for $\hat{c}_t(x_t, \eta_t)$

$$\hat{c}_t^{-\gamma}(x_t, \eta_t) = \beta e^{\Delta Z_{t+1}\theta}\ \pi_{t+1} E_t(R c_{t+1}^{-\gamma} g_{Y(t+1)} + \\ (1-\pi_{t+1})\beta\alpha(x_t - c_t)^{-\gamma} \tag{4}$$

in which I substituted the law of motion for x and y. In the presence of borrowing constraints, the consumption rule will then be given by

$$c_t(x_t, \eta_t) = \min\{\hat{c}_t(x_t, \eta_t), x_t\} \tag{5}$$

For x, I use 60 points between 0 and 5 and 40 between 5 and 40, and linearly interpolate for values between gridpoints. Smooth interpolation methods may be faster for most of the range of the function, but may be problematic near the borrowing constraint. For this reason, several gridpoints are necessary for low values of x. For η (a normally distributed variable), I use 5 gridpoints. To compute the expected value in (4) I also use numerical quadrature with the same number of points, so that no interpolation is necessary. After solving for the consumption rules c, I simulate the model for a large number of agents (10000).

The algorithm was implemented in Fortran 90 with IMSL routines. Note that because of the complexity of the function, for the estimation of β alone it is faster not to use an optimization routine based on derivatives. Derivatives are numerically evaluated only at the end, to compute the distribution of the estimator. For the estimation of β and γ, I first compute the value function on a fine grid, to locate the region of the minimum (since, as said, the function is rather flat), and then start an optimization routine from the minimum of the grid. While estimation of β alone takes 15 minutes on a desktop computer with a Pentium Pro processor, that of two parameters takes several hours.

The data used and the Fortran code can be found on my webpage at: http://www.people.virginia.edu/˜ mc6se/

References

[1991] Deaton, A. (1991), "Saving and Liquidity Constraints", *Econometrica*, 59, pp. 1221-1248.